Discretization of the Laplace Integral Operators

In the development of an appropriate boundary element method[1] for the solution of Laplace's Equation[2] the equation is first reformulated as an integral equation containing (the Laplace) integral operators[3] and the next objective is to rewrite the integral equations as a linear system of equations[4] and to do this the integral operators must be written in discrete form.

For all reformulations of the Laplace equations (a subset of) the same integral operators appear ($L, M, M^t$and N). Hence it makes sense to abstract this as a building block of the boundary element method, as, for example this has been carried out for a set of Fortran codes for the solutions of Helmholtz problems[5]. In this document the derivation of the discrete forms of the Laplace integral operators is outlined[6,7]. The simplest form of approximation in the derivation of the boundary element method is applied; the boundary is approximated by a set of panels and the boundary functions are approximated by a constant on each panel. A project for the development of codes for computing the discrete Laplace integral operators in Matlab, Excel VBA and Fortran is being implemented[8].

In order to derive the discrete forms of the integral operators *L, M,* M^t and *N* [3], the boundary or surface Γ is approximated by a set of *n* panels:

$$\Gamma \approx \tilde{\Gamma} = \sum_{j=1}^{n} \Delta\tilde{\Gamma}_j \, , \tag{1}$$

where the $\Delta\tilde{\Gamma}_j$ for $j = 1,2, \ldots n$ are the panels that together make up the boundary $\tilde{\Gamma}$, which is usually an approximation to the original boundary Γ.

Let us initially focus on one of the boundary integral operators – L - to illuminate the discretisation method, which can then be applied to the other operators. The L operator is defined as operating on an arbitrary function μ as follows[3]:

$$\{L\mu\}_\Gamma(\boldsymbol{p}) = \int_\Gamma G(\boldsymbol{p},\boldsymbol{q})\, \mu(\boldsymbol{q}) dS_q \, , \tag{2}$$

The boundary function μ is replaced by its equivalent $\tilde{\mu}$ on the approximate boundary $\tilde{\Gamma}$:

$$\{L\mu\}_\Gamma(\boldsymbol{p}) \approx \{L\tilde{\mu}\}_{\tilde{\Gamma}}(\boldsymbol{p}) = \int_{\tilde{\Gamma}} G(\boldsymbol{p},\boldsymbol{q})\, \tilde{\mu}(\boldsymbol{q}) dS_q \, . \tag{3}$$

The function is then replaced by a constant on each panel. Thus for the L operator:

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2 Laplace's Equation
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$$\int_{\tilde{\Gamma}} G(\boldsymbol{p},\boldsymbol{q})\,\tilde{\mu}(\boldsymbol{q})dS_q = \sum_{j=1}^{n}\int_{\Delta\widetilde{\Gamma_j}} G(\boldsymbol{p},\boldsymbol{q})\,\tilde{\mu}(\boldsymbol{q})dS_q \tag{4}$$

$$\approx \sum_{j=1}^{n}\int_{\Delta\widetilde{\Gamma_j}} G(\boldsymbol{p},\boldsymbol{q})\,\tilde{\mu}_{\boldsymbol{j}}dS_q = \sum_{j=1}^{n}\tilde{\mu}_{\boldsymbol{j}}\int_{\Delta\widetilde{\Gamma_j}} G(\boldsymbol{p},\boldsymbol{q})\,dS_q = \sum_{j=1}^{n}\tilde{\mu}_{\boldsymbol{j}}\,\{L\tilde{e}\}_{\Delta\widetilde{\Gamma_j}}(\boldsymbol{p})\,.$$

where e is the unit function. The other integral operators may be discretised in a similar way.

The discrete forms of the Laplace integral operators[3] are thus defined as follows:

$$\{Le\}_{\Delta\tilde{\Gamma}_j}(\boldsymbol{p}) = \int_{\Delta\tilde{\Gamma}_j} G(\boldsymbol{p},\boldsymbol{q})dS_q$$

$$\{Me\}_{\Delta\tilde{\Gamma}_j}(\boldsymbol{p}) = \int_{\Delta\tilde{\Gamma}_j} \frac{\partial G(\boldsymbol{p},\boldsymbol{q})}{\partial n_q}\,dS_q$$

$$\{M^t e\}_{\Delta\tilde{\Gamma}_j}(\boldsymbol{p};\,\boldsymbol{v}_p) = \frac{\partial}{\partial v_p}\int_{\Delta\tilde{\Gamma}_j} G(\boldsymbol{p},\boldsymbol{q})dS_q$$

$$\{Ne\}_{\Delta\tilde{\Gamma}_j}(\boldsymbol{p};\,\boldsymbol{v}_p) = \frac{\partial}{\partial v_p}\int_{\Delta\tilde{\Gamma}_j} \frac{\partial G(\boldsymbol{p},\boldsymbol{q})}{\partial n_q}\,dS_q$$

In the latter two definitions above the derivative can be taken inside the integral:

$$\{M^t e\}_{\Delta\tilde{\Gamma}_j}(\boldsymbol{p};\,\boldsymbol{v}_p) = \int_{\Delta\tilde{\Gamma}_j} \frac{\partial}{\partial v_p}\,G(\boldsymbol{p},\boldsymbol{q})\,\mu(\boldsymbol{q})\,dS_q$$

and

$$\{Ne\}_{\Delta\tilde{\Gamma}_j}(\boldsymbol{p};\,\boldsymbol{v}_p) = \int_{\Delta\tilde{\Gamma}_j} \frac{\partial^2 G(\boldsymbol{p},\boldsymbol{q})}{\partial v_p\,\partial n_q}\,dS_q\,,$$

except in the special case for the N_0 operator, when the observation point $\boldsymbol{p}$ lies on the panel $\Delta\widetilde{\Gamma_J}$, and in this case the evaluation of the discrete form requires special treatment.

When $\boldsymbol{p} \in \Delta\widetilde{\Gamma_J}$ then the integrand of the $\{L_0\,e'\}_{\Delta\widetilde{\Gamma_J}}(\boldsymbol{p})$ has a log r singularity at $\boldsymbol{p}$ in the two-dimensional case and a $\frac{1}{r}$ singularity at $\boldsymbol{p}$ in the three-dimensional case, and its evaluation also requires special treatment. Other than in these special cases, and provided $\boldsymbol{p}$ does not lie on the edge of a panel (where the normal is undefined) then in practice the integrands are regular and can be evaluated by standard quadrature[9].

[9] Numerical Integration or Quadrature