Integral Equation Formulations for the Exterior Laplace Problem

This chapter addresses the problem of solving the Laplace equation\(^1\) exterior to a closed surface or surfaces and subject to a boundary condition and to a specified incident field. Let the domain of the Laplace's equation be the region \(E\) exterior to the closed boundary \(S\), as illustrated in Figure 1.

![Fig 1. The domain of the exterior Laplace problem](image)

The problem is equivalent to the solution of the Laplace equation

\[
\nabla^2 \varphi(p) = 0 \quad (p \in E)
\]

in the domain \(E\). The boundary condition is assumed to take a general form

\[
\alpha(p) \varphi(p) + \beta(p) v(p) = f(p) \quad (p \in S)
\]

where \(\alpha\), \(\beta\) and \(f\) are real-valued functions defined on the boundary and \(v(p) = \frac{\partial \varphi(p)}{\partial n}\) and this completes the boundary-value problem\(^2\). Note that the normals to the boundary are taken to be in the outward direction.

Integral Equation Formulation

In this document we consider the integral equation formulations of the interior Laplace equation. An incident field along with a general boundary condition is included and this leads to more generalised boundary integral equations.

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\(^1\) Laplace’s Equation

\(^2\) Boundary-value Problems and Boundary Conditions
Direct Formulation

The application of Green's second theorem to the Laplace equation gives the following equations:

\[ \{ M \varphi \} (p) - \varphi (p) = \{ L v \} (p) \quad (p \in E), \]

\[ \{ M \varphi \} (p) - \frac{1}{2} \varphi (p) = \{ L v \} (p) \quad (p \in S), \]

where \( L \) and \( M \), the two of the Laplace integral operators\(^3\). The equation on \( S \), the boundary integral equation, may also be written in the following form

\[ \left\{ (M - \frac{1}{2} I) \varphi \right\} (p) = \{ L v \} (p) \quad (p \in S), \]

where \( I \) is the identity operator.

The above equations can be utilised to solve the interior Laplace equation by the boundary element method\(^4\) giving (approximations to) both \( \varphi \) and \( v \) on the boundary \( S \), using the second equation and then the first equation yields an approximation to \( \varphi (p) \) for any point \( p \) in the domain. There is only one small point with this approach and that is in the case of a Dirichlet boundary condition the equation on \( S \) is a Fredholm integral equation of the first kind, which should generally be avoided in numerical computation if possible, however, given that the kernel is singular the first kind equation is not expected to present any particular problem.

Differentiating each term of equation in \( E \) with respect to any vector \( v_p \) gives

\[ \frac{\partial}{\partial v_p} \{ M \varphi \} (p) - \frac{\partial \varphi (p)}{\partial v_p} = \frac{\partial}{\partial v_p} \{ L v \} (p) \quad (p \in E), \]

or

\[ \frac{\partial \varphi (p)}{\partial v_p} \{ N \varphi \} (p; v_p) - \frac{\partial \varphi (p)}{\partial v_p} = \frac{\partial}{\partial v_p} \{ M^t v \} (p; v_p) \quad (p \in D), \]

where \( M^t \) and \( N \) are also integral operators.

\(^3\) The Laplace Integral Operators

\(^4\) Introduction to the Boundary Element Method
By taking the limit as the point \( p \) approaches the boundary with the vector \( u_p \) being the unit outward normal to the boundary at \( p \) (that is \( n_p \)), and taking into account the jump properties of the \( M^e \) operator, the following boundary integral equation is obtained:

\[
\{ N \varphi \}_S (p; n_p) = \{ (M^e + - I) v \}_S (p; n_p) (p \in S).
\]

where \( v(p) = \partial \phi(p)/\partial n_p \), and this provides an alternative boundary integral equation on which the boundary element method could be based.

Another option is to base the boundary element method on a hybrid of the two direct boundary integral equations:

\[
\{ (M - - I + \mu N) \varphi \}_S (p; n_p) = \{ (L + \mu (M^e + - I)) v \}(p; n_p).
\]

Indirect Formulation

The indirect integral equation formulations are obtained by writing \( \varphi \) as a single or double layer potential;

\[
\varphi (p) = \{ L \sigma_0 \}_S(p) \quad \text{or} \quad \varphi (p) = \{ M \sigma_\infty \}_S(p) \quad (p \in E)
\]

where the \( \sigma_0 \) and \( \sigma_\infty \) are source density functions defined on \( S \). For points on the boundary the equations become boundary integral equations;

\[
\varphi (p) = \{ L \sigma_0 \}_S(p) \quad \text{or} \quad \varphi (p) = \{ (M+ - I) \sigma_\infty \}_S(p) \quad (p \in S)
\]

where the jump condition of \( M \) operator\(^5\) has been taken into account in the second equation.

Either of the two boundary integral equations can be used as the basis of the boundary element method. However, there is no \( \partial \varphi /\partial n \) term and hence, up to this point, only the Dirichlet problems can be solved. As with the direct method, a hybrid of the two equations can also be used as the basis of the boundary element method:

\[
\varphi (p) = \{ (L + \mu M) \sigma_\mu \}_S(p) \quad (p \in E),
\]

\[
\varphi (p) = \{ (L + \mu (M + - I)) \sigma_\mu \}_S(p) \quad (p \in S).
\]

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\(^5\) The Laplace Integral Operators
In order to also solve Neumann and Robin problems further equation(s) that involve the \( \frac{\partial \phi}{\partial n} \) term are required. Differentiating the above equations defined in \( E \) with respect to any vector \( \mathbf{v}_p \) gives the following equations

\[
\frac{\partial \phi(p)}{\partial \mathbf{v}_p} = \frac{\partial}{\partial \mathbf{v}_p} \{L \sigma_0\}_S(p) = \{M^t \sigma_0\}_S(p; \mathbf{v}_p) \quad (p \in E)
\]

and

\[
\frac{\partial \phi(p)}{\partial \mathbf{v}_p} = \frac{\partial}{\partial \mathbf{v}_p} \{M \sigma_0\}_S(p) = \{N \sigma_0\}_S(p; \mathbf{v}_p) \quad (p \in E).
\]

Taking the limit as the point \( p \) approaches a point on the boundary, and with \( \mathbf{v}_p = n_p \), gives the following boundary integral equations:

\[
\frac{\partial \phi(p)}{\partial n_p} = \{(M^t - \frac{1}{2} I) \sigma_0\}_S(p; n_p) \quad (p \in S)
\]

and

\[
\frac{\partial \phi(p)}{\partial n_p} = \{N \sigma_0\}_S(p; n_p) \quad (p \in S).
\]

which can be used for the solution of the Neumann problem.

Again, we have two equations that could form the basis of the boundary element method, and as an alternative, again a hybrid equation can be used:

\[
\frac{\partial \phi(p)}{\partial \mathbf{v}_p} = \{(M^t + \mu N) \sigma_\mu\}_S(p; \mathbf{v}_p) \quad (p \in E),
\]

which gives the following boundary integral equation

\[
\frac{\partial \phi(p)}{\partial n_p} = \{(M^t - \frac{1}{2} I + \mu N) \sigma_\mu\}_S(p; n_p) \quad (p \in S).
\]

One of the advantages in the indirect boundary element method is that the chosen boundary integral definitions can be substituted into the Robin boundary condition in order to obtain a generalised integral equation that represents the whole boundary value problem. For example if the single layer formulations are substituted into the Robin boundary condition then the following boundary integral equation results:

\[
\alpha(p) \{L \sigma_0\}_S(p) + \beta(p) \{(M^t - \frac{1}{2} I) \sigma_0\}_S(p; n_p) = f(p)
\]
Field Modification

As a further generalisation there may be an incident field in the domain, termed \( \varphi^i \), which is the field that would exist if there were no boundaries, or the free-space Laplace field. Such problems can also be solved by the boundary element method, it only requires a generalisation of the integral equations and the corresponding alteration of the resulting boundary element methods.

For the direct method, the solution of the exterior Laplace problem on the boundary \( S \) can be determined through first including the incident field in the direct formulation

\[
\varphi (p) = \varphi^i(p) + \{ M \varphi \}_S (p) - \{ L v \}_S (p) \quad (p \in E).
\]

Following the same technique, moving the point \( p \) to the boundary, gives the following boundary integral equation

\[
\left\{ \left( M - \frac{1}{2} I \right) \varphi \right\}_S (p) = -\varphi^i(p) + \{ L v \}_S (p) \quad (p \in S)
\]

In a similar fashion, the inclusion of the existing field term generalises the single layer potentials to give the following equations

\[
\varphi (p) = \varphi^i(p) + \{ L \sigma_0 \}_S (p) \quad (p \in S \cup E),
\]

\[
v(p) = v^i(p) + \left\{ \left( M^i - \frac{1}{2} I \right) \sigma_0 \right\}_S (p,n_p) \quad (p \in S)
\]

where \( v_i = \partial \varphi_i / \partial n_p \).