Numerical Solution of a General Linear System of Equations

In this document a numerical method for solving a general linear system of equations is developed. The problem can be most conveniently written in matrix-vector form as follows

$$A\underline{x} = B\underline{y} + \underline{c},$$

where A and B are known $n \times n$ matrices and $\underline{c}$ is a known n-vector with

$$\alpha_i\, x_i + \beta_i\, y_i = f_i \;\text{ for } i = 1 \dots n$$

where the α_i, β_i and f_i are constants with α_i and β_i are never both zero for each i. The vectors $\underline{x}$ and $\underline{y}$ is the solution of the process. Problems of this form occur for example in the boundary element method[1].

Clearly if all α_i are non-zero then the substitution $x_i = \frac{f_i}{\alpha_i} - \frac{\alpha_i}{\alpha_i}\, y_i$ for $i = 1 \dots n$ can transform the problem into the general linear system of equations problem[2]. This is similarly true if all the β_i are non-zero. However, in general, it is possible that some of the α_i and/or β_i are zero and hence the method employed must be able to adjust to this. As with all numerical methods, the potential for division by a 'small' number should also be avoided, where possible. Hence the general outline of the method is that we make substitutions for each x_i or y_i, that is highly dependent on the magnitude of α_i compared to β_i. Once the substitutions are made the system reduces to the standard linear system of equations that can be solved most conveniently, for example, by LU factorisation followed by forward and back substitution[3].

In order to consider the development of the method, let us look at an example of a system of equations of the above form of dimension 3:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

and

$$\alpha_1\, x_1 + \beta_1\, y_1 = f_1\,, \quad \alpha_2\, x_2 + \beta_2\, y_2 = f_2\,, \quad \alpha_3\, x_3 + \beta_3\, y_3 = f_3 \;.$$

In order to illustrate the method, let us assume that in this particular case that $|\beta_1| \gg |\alpha_1|$, $|\alpha_2| \gg |\beta_2|$ and $|\beta_3| \gg |\alpha_3|$, so that the following substitutions are proposed so that division by a zero or small number is avoided:

$$y_1 = \frac{f_1}{\beta_1} - \frac{\alpha_1}{\beta_1}\, x_1, \quad x_2 = \frac{f_2}{\alpha_2} - \frac{\beta_2}{\alpha_2}\, y_2, \quad y_3 = \frac{f_3}{\beta_3} - \frac{\alpha_3}{\beta_3}\, x_3 \;.$$

The system is prepared so that the second matrix-vector multiplication contains the terms to be substituted:

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[1] Boundary Element Method
[2] Solution of Linear Systems of Equations
[3] LU Factorisation of a Matrix

$$\begin{pmatrix} a_{11} & -b_{12} & a_{13} \\ a_{21} & -b_{22} & a_{23} \\ a_{31} & -b_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ y_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_{11} & -a_{12} & b_{13} \\ b_{21} & -a_{22} & b_{23} \\ b_{31} & -a_{32} & b_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ x_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

The expressions above are then substituted for y_1, x_2 and y_3 to give

$$\begin{pmatrix} a_{11} & -b_{12} & a_{13} \\ a_{21} & -b_{22} & a_{23} \\ a_{31} & -b_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ y_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_{11} & -a_{12} & b_{13} \\ b_{21} & -a_{22} & b_{23} \\ b_{31} & -a_{32} & b_{33} \end{pmatrix} \begin{pmatrix} \frac{f_1}{\beta_1} - \frac{\alpha_1}{\beta_1} x_1 \\ \frac{f_2}{\alpha_2} - \frac{\beta_2}{\alpha_2} y_2 \\ \frac{f_3}{\beta_3} - \frac{\alpha_3}{\beta_3} x_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

The terms in x_1, y_2 and x_3 are collected within the first matrix to give the following equation in standard form:

$$\begin{pmatrix} a_{11} - \frac{\alpha_1}{\beta_1} b_{11} & -b_{12} + \frac{\beta_2}{\alpha_2} a_{12} & a_{13} - \frac{\alpha_3}{\beta_3} b_{13} \\ a_{21} - \frac{\alpha_1}{\beta_1} b_{21} & -b_{22} + \frac{\beta_2}{\alpha_2} a_{22} & a_{23} - \frac{\alpha_3}{\beta_3} b_{23} \\ a_{31} - \frac{\alpha_1}{\beta_1} b_{31} & -b_{32} + \frac{\beta_2}{\alpha_2} a_{32} & a_{33} - \frac{\alpha_3}{\beta_3} b_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ y_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} \frac{b_{11}}{\beta_1} & -\frac{a_{12}}{\alpha_2} & \frac{b_{13}}{\beta_3} \\ \frac{b_{21}}{\beta_1} & -\frac{a_{22}}{\alpha_2} & \frac{b_{23}}{\beta_3} \\ \frac{b_{31}}{\beta_1} & -\frac{a_{32}}{\alpha_2} & \frac{b_{33}}{\beta_3} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

Once this system is solved the values of x_1, y_2 and x_3 will be determined. The values of y_1, x_2 and y_3 can then be found from the substitution formulae above.

Clearly if $|\beta_1| \gg |\alpha_1|$, $|\alpha_2| \gg |\beta_2|$ and $|\beta_3| \gg |\alpha_3|$ then the substitutions above are the correct ones to maintain and maximise numerical accuracy. However, what if α_i and β_i have a similar magnitude? If we look at the first column of the matrix on the left hand side above, it is the same as the column of A with $\frac{\alpha_1}{\beta_1}$ multiplied by the first column of B. If we look at the second column of the same matrix, this is made up of the second column of B and $\frac{\beta_2}{\alpha_2}$ multiplied by the first column of A. This suggests that the critical decision as to whether to substitute x_i or y_i should be dependent on the i-column-values of matrices A and B, as well as the values of α_i and β_i. Since the column values change from row to row then some general measure of the column values has to be used to automate this. It is suggested that the decision to substitute x_i or y_i is made using the following criterion: if $|\beta_i|^2 \|\underline{a}_i\| > |\alpha_i|^2 \|\underline{b}_i\|$ then the substitution $y_i = \frac{f_i}{\beta_i} - \frac{\alpha_i}{\beta_i} x_i$ is made, otherwise the substitution $x_i = \frac{f_i}{\alpha_i} - \frac{\beta_i}{\alpha_i} y_i$ is made, where $\|\_\_{i}\|$ denotes the vector norm of the i^{th} column of the matrix.

The system of equations above can normally be solved through LU factorisation and forward and back substitution[4]. Following a solution, note that a change in $\underline{f}$ and/or $\underline{c}$ alone only requires the re=application of forward and back substitution. Examples of computer programs for solving the general linear system and the follow-up solutions for new values of $\underline{f}$ and/or $\underline{c}$ are GLS[5] and REGLS[6] in FORTRAN, gls.m[7] and regls.m[8] in Matlab and gls.bas and regls.bas in GLS.xlsm[9].

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[4] LU Factorisation of a Matrix
[5] GLS.FOR
[6] REGLS.FOR
[7] gls.m
[8] regls.m
[9] GLS.xlsm