The Laplace Integral Operators

The Laplace integral operators arise when we reformulate Laplace's equation[1] as a boundary integral equation, which is an essential step in the development of the boundary element method[2]. Although various integral equations arise from various classes and dimensions of domain, the same set of integral operators tend to arise. Hence it is useful to consider these operators separately, as building blocks of integral formulations.

In this document the Laplace integral operators and a number of their properties are stated[3]. The operators are defined generally so that they can be adapted to the various Laplace problems. The notation is also useful in developing the boundary element method.

Definition of the Laplace integral operators

The Laplace integral operators are denoted L, M, M^t and N and are defined as follows:

$$\{L\mu\}_\Gamma(\boldsymbol{p}) = \int_\Gamma G(\boldsymbol{p},\boldsymbol{q})\,\mu(\boldsymbol{q})dS_q\,, \tag{1}$$

$$\{M\mu\}_\Gamma(\boldsymbol{p}) = \int_\Gamma \frac{\partial G(\boldsymbol{p},\boldsymbol{q})}{\partial n_q}\,\mu(\boldsymbol{q})dS_q\,, \tag{2}$$

$$\{M^t\mu\}_\Gamma(\boldsymbol{p};\boldsymbol{v}_p) = \frac{\partial}{\partial v_p}\int_\Gamma G(\boldsymbol{p},\boldsymbol{q})\,\mu(\boldsymbol{q})dS_q\,, \tag{3}$$

$$\{N\mu\}_\Gamma(\boldsymbol{p};\boldsymbol{v}_p) = \frac{\partial}{\partial v_p}\int_\Gamma \frac{\partial G(\boldsymbol{p},\boldsymbol{q})}{\partial n_q}\,\mu(\boldsymbol{q})dS_q\,. \tag{4}$$

where Γ is a boundary (not necessarily closed), $\boldsymbol{n}_q$ is the unique unit normal vector to Γ at $\boldsymbol{q}$, $\boldsymbol{v}_p$ is a unit directional vector passing through $\boldsymbol{p}$ and $\mu(\boldsymbol{q})$ is a function defined for $\boldsymbol{q} \in \Gamma$. In the definitions (1-4) the boundary Γ represents a line in a two-dimensional geometry and a surface in three-dimensional geometry and hence the integrals are line integrals[4] or surface integrals[5]. The operation of partial differentiation[6] $G(\boldsymbol{p},\boldsymbol{q})$ is a the free-space Green's function for the Laplace equation. The usual Green's functions are

$$G(\boldsymbol{p},\boldsymbol{q}) = -\frac{1}{2\pi}\ln r \quad \text{in two dimensions,} \tag{5}$$

$$G(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{4\pi r} \quad \text{in three dimensions.} \tag{6}$$

1 Laplace's Equation
2 www.boundary-element-method.com
3 The Boundary Element Method in Acoustics
4 Line Integration in Vector Calculus
5 Surface and Volume Integrals
6 Partial Differentiation

where $r = |\boldsymbol{r}|$, $\boldsymbol{r} = \boldsymbol{p}$-$\boldsymbol{q}$. Note the decaying behaviour of the three-dimensional Green's function. This property is conveyed to potential field produced from the surface in exterior problems. For the two-dimensional case, note the ln r behaviour of the solution as r approaches infinity; if this is not physically representative (for 'exterior' problems) then some modification or alternative Green's function would be required.

Properties of the Operators

In general for a given function $(\boldsymbol{p})$ $(\boldsymbol{p} \in \Gamma)$, $\{L\mu\}_\Gamma(\boldsymbol{p})$ and $\{N\mu\}_\Gamma(\boldsymbol{p};\boldsymbol{v}_p)$ are continuous across the boundary Γ (for any given unit vector $\boldsymbol{v}_p$ in the definition of the latter function). The $\{M\mu\}_\Gamma(\boldsymbol{p})$ and $\{M^t\mu\}_\Gamma(\boldsymbol{p};\boldsymbol{v}_p)$ operations are discontinuous at Γ and continuous on the remainder of the domain. The operators M and M^t have the following continuity properties at points in the neighbourhood of the boundary Γ :

$$\{M\,\zeta\}_\Gamma(\boldsymbol{p}+\varepsilon\boldsymbol{n}_p)+\frac{1}{2}\zeta(\boldsymbol{p}) = \{M\,\zeta\}_\Gamma(\boldsymbol{p}) = \{M\,\zeta\}_\Gamma(\boldsymbol{p}-\varepsilon\boldsymbol{n}_p)-\frac{1}{2}\zeta(\boldsymbol{p})\,, \tag{7}$$

$$\begin{aligned}\{M^t\,\zeta\}_\Gamma(\boldsymbol{p}+\varepsilon\boldsymbol{n}_p;\,\boldsymbol{n}_p)-\frac{1}{2}\zeta(\boldsymbol{p}) &= \{M^t\,\zeta\}_\Gamma(\boldsymbol{p};\,\boldsymbol{n}_p)\\ &= \{M^t\,\zeta\}_\Gamma(\boldsymbol{p}-\varepsilon\boldsymbol{n}_p;\,\boldsymbol{n}_p)+\frac{1}{2}\zeta(\boldsymbol{p})\end{aligned} \tag{8}$$

where $\boldsymbol{p} \in \Gamma$ and $\boldsymbol{n}_p$ is the unit normal to Γ at $\boldsymbol{p}$. The continuity properties are slightly different if Γ is not smooth at p, the $\frac{1}{2}s$ are replaced by $2\pi\,c(\boldsymbol{p})$ for two-dimensional problems and $4\pi\,c(\boldsymbol{p})$ for three-dimensional problems, where $c(\boldsymbol{p})$ is the (solid) angle subtended by the region at the boundary at $\boldsymbol{p}^+$ on the left equations and where $1 - c(\boldsymbol{p})$ is the (solid) angle subtended by the region at the boundary at p^- on the right equations.

Derivatives of G with respect to r

In two dimensions we have

$$\frac{\partial G}{\partial r} = -\frac{1}{2\pi}\frac{1}{r} \tag{9}$$

$$\frac{\partial^2 G}{\partial r^2} = \frac{1}{2\pi}\frac{1}{r^2} \tag{10}$$

,

In three dimensions we have

$$\frac{\partial G}{\partial r} = -\frac{1}{4\pi}\frac{1}{r^2} \tag{11}$$

$$\frac{\partial^2 G}{\partial r^2} = \frac{1}{2\pi}\frac{1}{r^3} \tag{12}$$

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Expressions for the normal derivatives of r

In this section we derive simpler expressions for the normal derivatives of G , that occur in the kernel functions in the definition of the integral operators (1-4). The simplifications require knowledge of the operations of vector geometry[7]. derivatives of G with respect to $\boldsymbol{v}_p$ and $\boldsymbol{n}_q$, may be written as follows:

$$\frac{\partial r}{\partial n_q} = \frac{\boldsymbol{r}.\boldsymbol{n}_q}{r} \tag{13}$$

$$\frac{\partial r}{\partial v_p} = \frac{\boldsymbol{r}.\boldsymbol{v}_p}{r} \tag{14}$$

$$\frac{\partial^2 r}{\partial v_p \partial n_q} = \frac{1}{r}\left(\boldsymbol{v}_p.\boldsymbol{n}_q + \frac{\partial r}{\partial v_p}\frac{\partial r}{\partial n_q}\right) \tag{15}$$

Expressions for the normal derivatives of G

With the results found above expressions for the kernel functions in the M, M^t, and N operators can be found:

$$\frac{\partial G}{\partial n_q} = \frac{\partial G}{\partial r}\frac{\partial r}{\partial n_q}, \tag{16}$$

$$\frac{\partial G}{\partial v_p} = \frac{\partial G}{\partial r}\frac{\partial r}{\partial v_p}, \tag{17}$$

$$\frac{\partial^2 G}{\partial v_p \partial n_q} = \frac{1}{2\pi r^2}\left(\boldsymbol{v}_p.\boldsymbol{n}_q + 2\frac{\partial r}{\partial v_p}\frac{\partial r}{\partial n_q}\right) \text{ in two dimensions and} \tag{18}$$

$$\frac{\partial^2 G}{\partial v_p \partial n_q} = \frac{1}{4\pi r^3}\left(\boldsymbol{v}_p.\boldsymbol{n}_q + 3\frac{\partial r}{\partial v_p}\frac{\partial r}{\partial n_q}\right) \text{ in three dimensions.} \tag{19}$$

[7] Vector Arithmetic and Geometry