

Jump Relations of Layer Potentials

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Abstract

Layer potentials are fundamental tools in the analysis and numerical solution of boundary value problems (BVPs)¹ for partial differential equations (PDEs)². The jump relations describe the discontinuities of these potentials and their derivatives across a boundary surface, providing key insights for boundary integral equation formulations. This article reviews the single- and double-layer potentials, and their associated jump relations, particularly for the Laplace equation³.

1 Introduction

In mathematical physics and PDE theory, **layer potentials** are integral representations of solutions to elliptic PDEs. Two classical types are:

- **Single-layer potentials**
- **Double-layer potentials**

These are constructed using the **fundamental solution** of the Laplace equations and a density function defined on the boundary of the domain.

The **jump relations** characterize the behaviour of these potentials and their normal derivatives as one approaches the boundary surface S from either side. These relations are essential in the boundary element method (BEM) and other integral equation approaches. In this article, the notation of the Laplace integral operators⁴ is also used.

2 Layer Potentials

2.1 Single-Layer Potential

For a density function $\sigma \in L^2(S)$ ⁵, the single-layer potential $L\sigma$ is defined as:

$$\{L\sigma\}(\mathbf{p}) = \int_S G(\mathbf{p}, \mathbf{q})\sigma(\mathbf{q}) dS_q, \quad \mathbf{p} \in \mathbb{R}^n \setminus S$$

¹Boundary Value Problems

²Partial Differential Equations (PDEs)

³Laplace's Equation

⁴The Laplace Integral Operators

⁵Function Spaces

This defines a harmonic function in $\mathbb{R}^n \setminus S$, continuous across S .

2.2 Double-Layer Potential

For a density $\sigma \in L^2(S)$, the double-layer potential $M\sigma$ is given by:

$$\{M\sigma\}(\mathbf{p}) = \int_S \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \sigma(\mathbf{q}) dS_q, \quad \mathbf{p} \in \mathbb{R}^n \setminus S$$

This potential is harmonic away from S , but **discontinuous across S** .

3 Jump Relations

Let $\mathbf{p}^* \in S$ be a point on the a smooth part of the boundary and let $\mathbf{n}_{p^*}$ be the unit outward normal to the boundary at $\mathbf{p}^*$. Let us now consider the continuity of the layer potentials and their derivatives as a point, close to the boundary, moves along the normal and through the boundary. it

3.1 Single-Layer Potential

- The potential $L\sigma$ is **continuous** across S :

$$\lim_{\epsilon \rightarrow 0} \{L\sigma(\mathbf{p} \pm \epsilon \mathbf{n}_p)\} = \{L\sigma\}(\mathbf{p})$$

- The **normal derivative** exhibits a jump:

$$\lim_{\epsilon \rightarrow 0} \frac{\partial(L\sigma)}{\partial n_p}(\mathbf{p} \pm \epsilon \mathbf{n}_p) = \frac{\partial(L\sigma)}{\partial n_p}(\mathbf{p}) \pm \frac{1}{2}\sigma(\mathbf{p})$$

as $\{M^t\sigma\}(\mathbf{p}) = \frac{\partial(L\sigma)}{\partial n_p}$ then the above limit can be written

$$\lim_{\epsilon \rightarrow 0} \{M^t\sigma\}(\mathbf{p} \pm \epsilon \mathbf{n}_p) = \{M^t\sigma\}(\mathbf{p}) \pm \frac{1}{2}\sigma(\mathbf{p})$$

3.2 Double-Layer Potential

- The potential has a **jump**:

$$\lim_{\epsilon \rightarrow 0} \{M\sigma\}(\mathbf{p} \pm \epsilon \mathbf{n}_p) = \{M\sigma\}(\mathbf{p}) \mp \frac{1}{2}\sigma(\mathbf{p})$$

where M is the boundary double-layer operator.

- The normal derivative of $M\sigma$ is **continuous** across S , assuming sufficient regularity of S and σ :

$$\lim_{\epsilon \rightarrow 0} \frac{\partial(M\sigma)}{\partial n_p}(\mathbf{p} \pm \epsilon \mathbf{n}_p) = \frac{\partial(M\sigma)}{\partial n_p}(\mathbf{p})$$

as $\{N\sigma\}(\mathbf{p}) = \frac{\partial(M\sigma)}{\partial n_p}$ then the above limit can be written

$$\lim_{\epsilon \rightarrow 0} \{N\sigma\}(\mathbf{p} \pm \epsilon \mathbf{n}_p) = \{N\sigma\}(\mathbf{p})$$

4 Conclusion

The jump relations of layer potentials are elegant and powerful tools that bridge PDE theory and numerical analysis. They clarify the behaviour of harmonic functions near boundaries and enable efficient reformulations of boundary value problems as integral equations.

References

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